

Period of the Simple Pendulum without Differential Equations

Alejandra Lira Vázquez^{a*}, Gulmaro Corona-Corona^b

^a*Universidad Nacional Autónoma de México. CCH-Azcapotzalco. Área de Matemáticas. Avenida Aquiles Serdán 2060, Azcapotzalco, Ex-hacienda el Rosario, CDMX. C.P. 02020, México*

^b*Universidad Autónoma Metropolitana, Unidad Azcapotzalco, Departamento de Ciencias Básicas e Ingeniería, Área de Matemáticas. Av. San Pablo, Colonia Reynosa Tamaulipas, Delegación Azcapotzalco, CDMX, C.P. 02200, México*

^a*Email: alv@unam.mx*

^b*Email: ccg@correo.azc.uam.mx*

Abstract

The purpose of this research is to learn that the topics of Physics, such as the period of a simple pendulum, only require the use of simple concepts of energy and trigonometric relationships to determine the theoretical mathematical model that allows to calculate this period, at the baccalaureate level. Two of the difficulties that are presented to students of High School Physics is to deal with complex equations for the determination of the period of a pendulum and the hard task of understanding which are the important variables in the mathematical model that determines the value of said period, that is why we decided to develop this work, so that it serves the Physics teacher as a strategy in the classroom in the development of the simple pendulum theme without Differential Equations.

Keywords: Period; period of the simple pendulum; Period without differential equations.

1. Introduction

In a pendulum the period is the time necessary to perform an oscillation. To determine the expression to calculate the period of a simple pendulum at any angle, we only have to establish the relations of the variables that intervene in the pendulum movement.

* Corresponding author.

Some authors like in [2, 3, 5, 6, 7, 8] establish differential equations for such movement. In this work we deduce the expression of the period of the simple pendulum, only using body diagram of the simple pendulum, the conservation of energy, the definition of speed, the period and the trigonometric functions, so that it is understandable by High School students.

2. Materials and Methods

The period of the simple pendulum is calculated without the use of differential equations, which makes it possible to have an explanation of this topic for High School students. Only the free body diagram of the simple pendulum and the trigonometric relations of a right triangle are used, considering the law of conservation of energy and some relations of basic mechanical properties.

3. Results

If we consider the movement of a pendulum, we can establish the mathematical model to calculate its period without using differential equations.

To obtain the period of a simple pendulum at any angle, we only have to establish the relationships of the variables that intervene in the pendulum movement, referring to Figure 1.

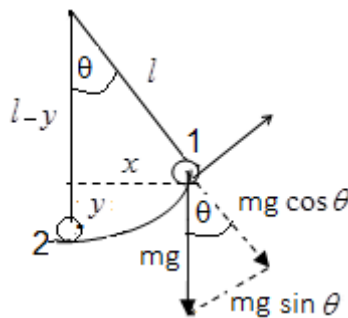


Figura 1

$$\sin \theta = \frac{x}{l} \quad (1)$$

$$\cos \theta = \frac{l-y}{l} \quad (2)$$

Solving for y, it has:

$$y = l(1 - \cos \theta) \quad (3)$$

From figure 1, we can observe the formation of a right triangle with sides $l - y$, $l y$ x .

Where:

l : Length of the pendulum.

x : Distance traveled horizontally.

y : Height traveled by the particle along the vertical axis.

θ : Change in the angle that the length of the pendulum forms with

regarding its initial position 1.

2: end point of the movement.

1: starting point of the movement.

In this type of movement, the law of conservation of energy is applicable, since there are no external forces acting on the body, therefore the change in kinetic energy is equal to the change in potential energy, which establishes the following relationships:

$$\Delta E_p = \Delta E_c$$

The change in potential energy being equal to:

$$\Delta E_p = mg(y - y_0)$$

and the change in kinetic energy equal to:

$$\Delta E_c = \frac{1}{2}m(v^2 - v_0^2)$$

Because $y_0 = 0$ y $v_0 = 0$

Then:

$$\Delta E_p = mgy$$

$$\Delta E_c = \frac{1}{2}mv^2$$

So by matching the change in kinetic energy with the change in potential energy we have to:

$$mgy = \frac{1}{2}mv^2 \quad (4)$$

Where:

ΔE_p : change in potential energy.

ΔE_c : change in kinetic energy.

M: mass of the particle in the pendulum.

g: acceleration due to gravity.

y: height that the particle travels along the vertical axis.

v: speed of the particle.

Solving for the speed of the equation (4) we get

$$v^2 = \frac{2mgy}{m}$$

When simplifying we have:

$$v^2 = 2gy \quad (\text{does not depend on the mass})$$

$$v = \sqrt{2gy} \quad (\text{depends on the height at which the pendulum arrives})$$

Substituting the relationship (3), we have:

$$v = \sqrt{2gl(1 - \cos\theta)} \quad (5)$$

From Kinematics, we establish the formula of the average speed of a particle:

$$v = \frac{d}{t} \quad (6)$$

Where:

d : distance traveled over the trajectory of the pendulum particle, the arc length between position 1 and 2.

t : time the movement last.

From figure 1, it can be seen that the trajectory that follows the particle of the pendulum is the length of the arc, which is given by:

$$d = l\theta$$

Substituting d in equation (6), we obtain:

$$v = \frac{l\theta}{t} \quad (7)$$

Equating equation (5) with (7) we have:

$$\sqrt{2g} \sqrt{l(1 - \cos\theta)} = \frac{l\theta}{t} \quad (8)$$

Solving for t from this equation, we obtain:

$$t = \frac{l\theta}{\sqrt{2g} \sqrt{l(1 - \cos\theta)}}$$

Rearranging terms we have to:

$$t = \frac{l\theta}{\sqrt{2gl} \sqrt{(1 - \cos\theta)}} \quad (9)$$

Considering the approximation for $\cos\theta$ (for small angles) and substituting in equation (9), we obtain:

$$t = \frac{l\theta}{\sqrt{2gl} \sqrt{(1 - (1 - \frac{1}{2}\theta^2))}}$$

Making the 3 following operations:

$$t = \frac{l\theta}{\sqrt{2gl} \sqrt{\frac{1}{2}\theta^2}}$$

$$t = \frac{l\theta}{\sqrt{gl} \sqrt{(2) \frac{1}{2}\theta^2}}$$

$$t = \frac{l\theta}{\sqrt{gl}\theta}$$

Simplifying θ we obtain:

$$t = \frac{l}{\sqrt{g} \sqrt{l}} \quad (10)$$

Multiplying \sqrt{l} by the numerator and denominator of equation (10), we have:

$$t = \frac{l}{\sqrt{g}} \frac{\sqrt{l}}{l}$$

Simplifying the lengths, we have that:

$$t = \sqrt{\frac{l}{g}} \quad (11)$$

As the period of the simple pendulum is equal to:

$$T = 2\pi t$$

Where:

T: Pendulum period.

t: Time the movement lasts.

Then the period of a simple pendulum is obtained from:

$$T = 2\pi \sqrt{\frac{l}{g}}$$

4. Discussion

Although in reference [1], does not use differential equations in the deduction of the equation to calculate the period of the simple pendulum and the same expression is reached, it presents differences, with this work, since the author uses the concept of frequency and the relations between the forces present in the pendulum movement. On the other hand, in reference [2], there is a difference with this work because it uses differential equations and the relationship of frequency with period, to arrive at the equation of the period of the simple pendulum. With respect to the reference [3], it presents great differences, with our work, since it considers differential equations and the concept of periodic function, when increasing the $2\pi \text{ rad}$ phase in a time T, to establish the expression to calculate the period. The reference [4], has a difference with our work, because it does not deduce the expression, only takes it for granted. Reference [5] has differences because it considers differential equations and the definition of simple harmonic motion, as well as the definition of periodic

function, since it establishes that after a time $T = \frac{2\pi}{\omega}$, the function is repeated. The references [6,7,8] have several differences with our work, because they use differential equations and they manage to establish the expression to calculate the period of the simple pendulum considering an approximation, also consider relations between the force, angular speed and the time.

5. Conclusions

Although, there are other ways to get to the expression of the period of the simple pendulum, the way to deduce the equation of the period of the pendulum in this work allows teachers to induce High School students, in a simple way, using Trigonometry and the law of the conservation of energy primarily, to conclude that the period does not depend on the mass or the angle or the height to which it is released, it only depends on its length and the acceleration of the gravity of the place where the movement was made, without using differential equations.

Therefore, to deduce the expression of the period of a simple pendulum, in theoretical form, without differential equations, we must consider that:

- 1) The change in kinetic energy is equal to the change in potential energy.
- 2) The height at which the pendulum is located is determined with a trigonometric function, with an approximation for small angles.
- 3) The speed is the quotient of the length of the arc traveled and the time of movement, respectively.

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